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A VARIATIONAL APPROACH TO THE THEORY OF THE ELASTIC BEHAVIOUR OF MULTIPHASE MATERIALS*

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Summary

Variational principles in the linear theory of elasticity, involving the elastic polarization tensor, have been applied to the derivation of upper and lower bounds for the effective elastic moduli of quasi-isotropic and quasi-homogeneous multiphase materials of arbitrary phase geometry. When the ratios between the different phase moduli are not too large the bounds derived are close enough to provide a good estimate for the effective moduli. Comparison of theoretical and experimental results for a two-phase alloy showed good agreement.

1. Introduction

The present work is concerned with the derivation of bounds for the effective elastic moduli of multiphase materials, with the aid of some variational principles in elasticity which have been previously established by the authors (Hashin and Shtrikman 1961 a, b, 1962).

The materials here considered may be described as mechanical mixtures of a number of different isotropic and homogeneous elastic phases. Assuming that, in the large, such a material can be regarded as quasi-isotropic and quasi-homogeneous, the problem is to predict the effective elastic moduli of the multiphase material in terms of the elastic moduli and volume fractions of the constituting phases.

The effective elastic moduli of a quasi-isotropic and quasi-homogeneous composite material may be conveniently defined by means of the strain energy stored in the material when subjected to gross uniform strains or stresses [compare Hashin (1962) and Section 3 below]. In order to evaluate this strain energy it is necessary to find the stress or displacement fields in the composite body, which appears to be an impossible task. Consequently, attempts to find *expressions* for effective elastic moduli, or other physical constants, have invariably been based on simplify-

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ing assumptions, concerning geometrical form and physical behaviour of phase regions.

A more attractive approach consists of the use of variational principles in order to bound the strain energy and thus also the effective elastic moduli. The principles of minimum potential energy and minimum complementary energy have been used in such a way by PAUL (1960) and HASHIN (1962). While Paul's treatment is based on arbitrary phase geometry, the bounds obtained are generally not close enough to provide a good estimate for the effective elastic moduli.

The bulk modulus bounds obtained by Hashin coincided whereas the shear modulus bounds are much closer than those obtained by Paul. However, the results are strictly valid only for a material consisting of a matrix in which certain special distributions of spherical inclusions are embedded.

The purpose of the present work is to find improved bounds without making assumptions about phase geometry.

2. VARIATIONAL PRINCIPLES

Some new variational principles for non-homogeneous linear elasticity, in terms of the elastic polarization tensor, have been formulated and proved by HASHIN and SHTRIKMAN (1961b) for the isotropic case and by HASHIN and SHTRIKMAN (1962) for the general anisotropic case. For the present purpose of derivation of bounds for the effective elastic moduli of multiphase materials, consisting of isotropic phases, the variational principles for non-homogeneous and isotropic elasticity, in the case of prescribed surface displacements, are needed. These will be formulated in the following.

Let σ°_{ij} and ϵ°_{ij} be known stress and strain tensor fields in a deformed elastic body of volume V and surface S. For simplicity the case of no body forces is considered. Hooke's law is given by

$$\sigma^\circ_{ij} = \lambda_0\, \epsilon^\circ_{kk}\, \delta_{ij} + 2G_0\, \epsilon^\circ_{ij} = L_0\,(\epsilon^\circ_{ij}). \tag{2.1}$$

Here λ_0 and G_0 are the Lamé constant and the shear modulus which for simplicity are taken to be constant throughout the body; the range of subscripts is 1, 2, 3; a repeated subscript denotes summation and δ_{ij} are the Kronecker delta. The strains are given in terms of the displacements by

$$\epsilon^\circ_{ij} = \tfrac{1}{2}\,(u^\circ_{i,j} + u^\circ_{j,i}) \tag{2.2}$$

where a comma denotes differentiation.

Let part or whole of the body now be changed to material of different moduli λ and G which may vary in space and the surface displacements $u^\circ_i\,(S)$ be held fixed. The unknown stress and strain fields in the new body are denoted by σ_{ij} and ϵ_{ij}.

The stress polarization tensor p_{ij}* is now defined by

$$\sigma_{ij} = L_0\,(\epsilon_{ij}) + p_{ij} \tag{2.3}$$

where L_0 is given by (2.1). Define also

$$u'_i = u_i - u^\circ_i \tag{2.4}$$

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*The polarization tensor has been implicitly introduced into the theory of elasticity by ESHELBY (1957). The name has been coined by KRÖNER (1958) in analogy to the polarization vector in electrostatics.

and consequently

$$\epsilon'_{ij} = \epsilon_{ij} - \epsilon^{\circ}_{ij}. \tag{2.5}$$

The ϵ_{ij} and σ_{ij} can be found from (2.5) and (2.3) once ϵ'_{ij} and p_{ij} are known.

Variational principles involving ϵ'_{ij} and p_{ij} will now be formulated. The volume integral

$$U_p = U_0 - \tfrac{1}{2} \int \left[p_{ij} \, H(p_{ij}) - p_{ij} \, \epsilon'_{ij} - 2p_{ij} \, \epsilon^{\circ}_{ij} \right] dV \tag{2.6}$$

where

$$U_0 = \tfrac{1}{2} \int \sigma^{\circ}_{ij} \, \epsilon^{\circ}_{ij} \, dV, \tag{2.7}$$

subject to the subsidiary condition

$$L_0(\epsilon'_{ij})_{,j} + p_{ij,j} = 0 \tag{2.8}$$

and the boundary condition

$$u'_i(S) = 0, \tag{2.9}$$

is stationary for

$$p_{ij} = L(\epsilon_{ij}) - L_0(\epsilon_{ij}). \tag{2.10}$$

The expression $L(\epsilon_{ij})$ in (2.10) is given by

$$L(\epsilon_{ij}) = \lambda \epsilon_{kk} \, \delta_{ij} + 2G\epsilon_{ij}. \tag{2.11}$$

It follows from (2.3) and (2.11) that the extremum condition (2.10) is equivalent to Hooke's law for the nonhomogeneous body.

The operator H in (2.6) is given by

$$H = (L - L_0)^{-1}. \tag{2.12}$$

Accordingly, in (2.6),

$$p_{ij} \, H(p_{ij}) = -\frac{\lambda - \lambda_0}{6\,(G - G_0)\,(K - K_0)} \, p^2_{kk} + \frac{1}{2\,(G - G_0)} \, p_{ij} \, p_{ij} \tag{2.13}$$

where

$$K = \lambda + \tfrac{2}{3} G$$

is the bulk modulus.

Equation (2.8) expressed in terms of u'_i assumes the form

$$(\lambda_0 + G_0) \, u'_{j,ji} + G_0 \, u'_{i,jj} + p_{ij,j} = 0. \tag{2.14}$$

It should be noted that (2.4), (2.5), (2.9), (2.10) and (2.14) are equivalent to the second boundary value problem of the theory of elasticity (SOKOLNIKOFF 1956).

The stationary value of U_p is an absolute maximum when

$$\lambda > \lambda_0, \quad G > G_0, \tag{2.15}$$

and an absolute minimum when

$$\lambda < \lambda_0, \quad G < G_0. \tag{2.16}$$

Furthermore, the stationary value of U_p is equal to the strain energy U stored in the changed body.

In the extreme case of vanishing K_0 and G_0 (in comparison to K and G), the variational principle reduces in the limit to the principle of minimum complementary energy. For infinitely large K_0 and G_0 the principle of minimum potential energy is obtained [for proof of the preceding results in the general anisotropic case, see HASHIN and SHTRIKMAN (1962)].

For prescribed surface tractions different variational principles, involving the *strain polarization* tensor, apply (HASHIN and SHTRIKMAN 1962).

3. BOUNDS FOR THE EFFECTIVE MODULI OF MULTIPHASE MATERIALS

Consider a composite body which consists of n different elastic phases and which may be regarded as quasi-homogeneous and quasi-isotropic. Let a surface displacement of the form

$$u^\circ_i\,(S) = \epsilon^\circ_{ij}\, x_j \tag{3.1}$$

be impressed on the composite body, where the ϵ°_{ij} are constants and the x_j are cartesian co-ordinates, referred to a fixed system of axes. Then it can be proved that the mean strains in the composite body are ϵ°_{ij}.

The assumed quasi-homogeneity of the multiphase material will here be interpreted in the following way : consider *any* reference cube in the composite material which is large compared to the size of non-homogeneities, yet small compared to the whole body. Then the volume average of a quantity such as displacement, strain, stress or phase volume fraction is the same for the whole body and the reference cube. It follows that the mean strains in any reference cube are also taken as ϵ°_{ij}.

To define the effective elastic moduli it is assumed that the elastic strain energy in a reference cube of unit volume can be represented in the form

$$U = \tfrac{1}{2}\,(9K^*\,\epsilon^{\circ 2} + 2G^*\, e^\circ_{ij}\, e^\circ_{ij}) \tag{3.2}$$

where the mean strains ϵ°_{ij} have been split into isotropic and deviatoric parts as follows :

$$\epsilon^\circ_{ij} = \epsilon^\circ\,\delta_{ij} + e^\circ_{ij} \tag{3.3}$$

where

$$\epsilon^\circ = \tfrac{1}{3}\,\epsilon_{kk}$$

and K^* and G^* are the effective bulk and shear moduli, respectively.

The variational formulation given above will now be applied to the reference cube of unit volume. The ϵ°_{ij} and ϵ'_{ij} there defined are then given the following interpretation with regard to the composite body. Assume that the surface displacements (3.1) are impressed on a homogeneous body whose elastic moduli are K_0 and G_0. It follows from the theory of elasticity that the strains throughout this body are constant and equal to ϵ°_{ij}. When (3.1) is prescribed on the surface of a composite body the strains in any cube of reference are

$$\epsilon_{ij} = \epsilon^\circ_{ij} + \epsilon'_{ij}. \tag{3.4}$$

Accordingly, the ϵ°_{ij} are identified with the mean stiains and the ϵ'_{ij} with the deviations from the mean. It follows that

$$\bar{\epsilon}'_{ij} = 0 \tag{3.5}$$

where the bar denotes mean value within the reference cube.

Expression (2.6) will now be evaluated for the reference cube and for this purpose the following polarization field p_{ij} will be chosen :

$$p_{ij} = p^r_{ij} \text{ in } V_r, \tag{3.6}$$

where V_r is the volume of the r^{th} phase and ${p^r}_{ij}$ is constant throughout this volume*. For convenience the volume fractions v_r are defined by

$$v_r = V_r/V \tag{3.7}$$

where V is the total volume. It follows that

$$\sum_{r=1}^{r=n} v_r = 1. \tag{3.8}$$

Since the reference cube is of unit volume the V_r in it can be replaced by the v_r.

Splitting all tensors in (2.6) into isotropic and deviatoric parts, this integral can be rewritten in the form

$$U_p = U_0 + U' - \tfrac{1}{2} \int \left(\frac{p^2}{K - K_0} + \frac{f_{ij} f_{ij}}{2\,(G - G_0)} - 6p\epsilon^\circ - 2f_{ij}\, {e^\circ}_{ij} \right) dV \tag{3.9}$$

where

$$U_0 = \tfrac{1}{2}\,(K_0\, \epsilon^{\circ 2} + 2G_0\, {e^\circ}_{ij}\, {e^\circ}_{ij}) \tag{3.10}$$

$$U' = \tfrac{1}{2} \int (3p\epsilon' + f_{ij}\, {e'}_{ij})\, dV \tag{3.11}$$

and f_{ij} and ${e'}_{ij}$ are the deviatoric parts, $p\delta_{ij}$ and $\epsilon'\, \delta_{ij}$ the isotropic parts, of p_{ij} and ${\epsilon'}_{ij}$ respectively. Introducing (3.6) into (3.9) and using (3.5) yields the result

$$U_p = U_0 + U' - \tfrac{1}{2} \sum_{r=1}^{r=n} \left[\frac{(p^r)^2}{K_r - K_0} + \frac{{f^r}_{ij}\, {f^r}_{ij}}{2\,(G_r - G_0)} \right] v_r + 3\bar{p}\epsilon^\circ + \bar{f}_{ij}\, {e^\circ}_{ij} \tag{3.12}$$

where $\bar{p}$ and $\bar{f}_{ij}$ are the mean values of p and f_{ij}, respectively, which on using (3.6) are found to be

$$\bar{p} = \sum_{r=1}^{r=n} p^r\, v_r, \tag{3.13}$$

$$\bar{f}_{ij} = \sum_{r=1}^{r=n} {f^r}_{ij}\, v_r. \tag{3.14}$$

It remains to evaluate U' given by (3.11) in terms of the polarization field (3.6). This can be formally done by Fourier methods, making use of (2.9) and (2.14). Some details of the derivation are given in the Appendix. The result is

$$2U' = \alpha_0 \left[\sum_{r=1}^{r=n} (p^r)^2\, v_r - \bar{p}^2 \right] + \beta_0 \left[\sum_{r=1}^{r=n} {f^r}_{ij}\, {f^r}_{ij}\, v_r - \bar{f}_{ij}\, \bar{f}_{ij} \right] \tag{3.15}$$

where

$$\alpha_0 = -\frac{3}{3K_0 + 4G_0}, \tag{3.16}$$

$$\beta_0 = -\frac{3\,(K_0 + 2G_0)}{5G_0\,(3K_0 + 4G_0)}. \tag{3.17}$$

Introducing (3.15) into (3.12) one obtains

$$\begin{aligned} U_p = U_0 - \tfrac{1}{2} \sum_{r=1}^{r=n} & \left[\frac{(p^r)^2}{K_r - K_0} - \alpha_0\,(p^r)^2 + \frac{{f^r}_{ij}\, {f^r}_{ij}}{(G_r - G_0)} - \beta_0\, {f^r}_{ij}\, {f^r}_{ij} \right] v_r \\ & - \tfrac{1}{2}\alpha_0\, \bar{p}^2 + 3\bar{p}\epsilon^\circ - \tfrac{1}{2}\beta_0\, \bar{f}_{ij}\, \bar{f}_{ij} + \bar{f}_{ij}\, {\epsilon^\circ}_{ij}. \end{aligned} \tag{3.18}$$

***It can be shown that restriction to a piecewise constant polarization field is not necessary. The subsequent analysis, yielding the same results, can be carried out for a polarization field restricted only by the assumption of isotropic space distribution of each component (see Appendix for use of this assumption).**

It follows from the maximum condition (2.15) that whenever K_0 and G_0 are such that, for *every* K_r and G_r,

$$K_r > K_0, \quad G_r > G_0, \tag{3.19}$$

then U_p satisfies the inequality

$$U_p < U. \tag{3.20}$$

Analogously, from (2.16), whenever K_0 and G_0 satisfy the inequalities

$$K_r < K_0, \quad G_r < G_0, \tag{3.21}$$

then

$$U_p > U. \tag{3.22}$$

Taking U in the form (3.2) and using (3.18), inequalities (3.20) and (3.22) become bounds on the effective elastic moduli. In order to find the best bounds for a polarization field of type (3.6), (3.18) will be maximized for condition (3.20) and minimized for condition (3.22). Differentiating (3.18) with respect to p^r and $f^r{}_{ij}$, the extremum conditions are found to be

$$-\frac{p^r}{K_r - K_0} + \alpha_0 p^r - \alpha_0 \bar{p} + \beta\epsilon^\circ = 0, \tag{3.23}$$

$$-\frac{f_{ij}{}^r}{2\,(G_r - G_0)} + \beta_0 f^r{}_{ij} - \beta_0 \bar{f}_{ij} + e^\circ{}_{ij} = 0. \tag{3.24}$$

These are found to be maximum conditions when (3.19) holds and minimum conditions for (3.21).

Introducing (3.23) and (3.24) into (3.18), this assumes the simple form

$$U_p = U_0 + \tfrac{1}{2}\,(3\bar{\tilde{p}}\epsilon^\circ + \bar{\tilde{f}}_{ij}\, e^\circ{}_{ij}) \tag{3.25}$$

where $\tilde{p}$ and $\tilde{f}_{ij}$ are solutions of (3.23) and (3.24). Expression (3.23) can also be written in the form

$$U_p = U_0 + \tfrac{1}{2}\,\bar{\tilde{p}}_{ij}\,\epsilon_{ij}. \tag{3.26}$$

The mean values $\bar{\tilde{p}}$ and $\bar{\tilde{f}}_{ij}$ can be simply determined from (3.23) and (3.24) by solving for p^r and $f^r{}_{ij}$ and using (3.13) and (3.14). The results are

$$\bar{\tilde{p}} = \epsilon^\circ \frac{3A}{1 + \alpha_0 A}, \tag{3.27}$$

$$\bar{\tilde{f}}_{ij} = e^\circ{}_{ij} \frac{B}{1 + \beta_0 B}, \tag{3.28}$$

where

$$A = \sum_{r=1}^{r=n} \frac{v_r}{\dfrac{1}{K_r - K_0} - \alpha_0}, \tag{3.29}$$

$$B = \sum_{r=1}^{r=n} \frac{v_r}{\dfrac{1}{2\,(G_r - G_0)} - \beta_0}. \tag{3.30}$$

In order to find bounds for the effective bulk modulus let a mean strain system of the form

$$\epsilon^\circ{}_{ij} = \epsilon^\circ\,\delta_{ij} \tag{3.31}$$

be applied to the composite body. It follows from (3.2), (3.20), (3.22), (3.25) and (3.27) that

$$K^* \gtrless K_0 + \frac{A}{1 + \alpha_0 A} \tag{3.32}$$

where the upper inequality sign applies for the first condition (3.19) and the lower for the first condition (3.21).

Similarly, when ϵ°_{ij} is of purely deviatoric form,

$$\epsilon^\circ_{ij} = e^\circ_{ij}, \quad e^\circ_{kk} = 0, \tag{3.33}$$

then from (3.2), (3.20), (3.22), (3.25) and (3.28),

$$G^* \gtrless G_0 + \tfrac{1}{2}\frac{B}{1 + \beta_0 B} \tag{3.34}$$

where the inequality signs apply in turn for the second conditions (3.19) and (3.31). It remains to specify the values of K_0 and G_0 which when introduced into (3.32) and (3.34) will yield the highest lower bounds and the lowest upper bounds. It can be proved by differentiation that the expressions in the right sides of (3.32) and (3.34) are monotonically increasing functions of K_0 and G_0. Accordingly, the highest lower bounds are obtained by taking the largest values of K_0 and G_0 which comply with (3.19). The lowest upper bounds are obtained for the smallest values of K_0 and G_0 which comply with (3.21). Let the smallest of the moduli K_r and G_r be denoted by K_1 and G_1 and the largest by K_n and G_n. Then for the highest lower bounds, which are denoted by K^*_1 and G^*_1,

$$K_0 = K_1, \quad G_0 = G_1, \tag{3.35}$$

and for the lowest upper bounds K^*_2 and G^*_2

$$K_0 = K_n, \quad G_0 = G_n. \tag{3.36}$$

Introducing in turn (3.35) and (3.36) into the right side of (3.32) it is found that

$$K^*_1 = K_1 + \frac{A_1}{1 + \alpha_1 A_1}, \tag{3.37}$$

$$K^*_2 = K_n + \frac{A_n}{1 + \alpha_n A_n}, \tag{3.38}$$

where from (3.16)

$$\alpha_1 = -\frac{3}{3K_1 + 4G_1}, \tag{3.39}$$

$$\alpha_n = -\frac{3}{3K_n + 4G_n}, \tag{3.40}$$

and from (3.29)

$$A_1 = \sum_{r=2}^{r=n} \frac{v_r}{\dfrac{1}{K_r - K_1} - \alpha_1}, \tag{3.41}$$

$$A_n = \sum_{r=1}^{r=n-1} \frac{v_r}{\dfrac{1}{K_r - K_n} - \alpha_n}, \tag{3.42}$$

and

$$K^*_1 < K^* < K^*_2. \tag{3.43}$$

Introducing in turn (3.35) and (3.36) into the right side of (3.34) it is found that

$$G^*_1 = G_1 + \tfrac{1}{2} \frac{\beta_1}{1 + \beta_1 B_1}, \tag{3.44}$$

$$G^*_2 = G_n + \tfrac{1}{2} \frac{B_n}{1 + \beta_n B_b}, \tag{3.45}$$

where from (3.17)

$$\beta_1 = - \frac{3 (K_1 + 2G_1)}{5G_1 (3K_1 + 4G_1)}, \tag{3.46}$$

and from (3.30)

$$\beta_n = - \frac{3 (K_n + 2G_n)}{5G_n (3K_n + 4G_n)}, \tag{3.47}$$

$$B_1 = \sum_{r=2}^{r=n} \frac{v_r}{\dfrac{1}{2 (G_r - G_1)} - \beta_1}, \tag{3.48}$$

$$B_n = \sum_{r=1}^{r=n-1} \frac{v_r}{\dfrac{1}{2 (G_r - G_n)} - \beta_n}, \tag{3.49}$$

and

$$G^*_1 < G^* < G^*_2. \tag{3.50}$$

An interesting result is found on evaluation of the bounds (3.32) and (3.34) for the extreme cases of vanishingly small or infinitely large K_0 and G_0. It is found by a limiting process that the lower bounds are given by

$$\frac{1}{K^*_0} = \sum_{r=1}^{r=n} \frac{v_r}{K_r}, \tag{3.51}$$

$$\frac{1}{G^*_0} = \sum_{r=1}^{r=n} \frac{v_r}{G_r}, \tag{3.52}$$

and the upper bounds are given by

$$K^*_\infty = \sum_{r=1}^{r=n} K_r v_r, \tag{3.53}$$

$$G^*_\infty = \sum_{r=1}^{r=n} G_r v_r. \tag{3.54}$$

The bounds (3.51) and (3.52) can be directly obtained from the principle of minimum complementary energy and (3.53) and (3.54) from the principle of minimum potential energy (PAUL 1960). This checks with the reduction of the present variational principles to those of minimum complementary and potential energy for vanishing and infinitely large K_0 and G_0. It is interesting to note that (3.51), (3.52), (3.53) and (3.54) are the worst bounds that can be derived by the present theory (assuming positive K_0 and G_0).

4. APPLICATION TO TWO-PHASE MATERIALS

The results obtained in the previous Section will now be specialized to the important case of two-phase materials. Expressions (3.37), (3.38), (3.44) and (3.45) have simply to be written out for $n = 2$. The results are

$$K^*_1 = K_1 + \frac{v_2}{\dfrac{1}{K_2 - K_1} + \dfrac{3v_1}{3K_1 + 4G_1}}, \tag{4.1}$$

$$K^*_2 = K_2 + \frac{v_1}{\dfrac{1}{K_1 - K_2} + \dfrac{3v_2}{3K_2 + 4G_2}}, \tag{4.2}$$

$$G^*_1 = G_1 + \frac{v_2}{\dfrac{1}{G_2 - G_1} + \dfrac{6(K_1 + 2G_1)v_1}{5G_1(3K_1 + 4G_1)}}, \tag{4.3}$$

$$G^*_2 = G_2 + \frac{v_1}{\dfrac{1}{G_1 - G_2} + \dfrac{6(K_2 + 2G_2)v_2}{5G_2(3K_2 + 4G_2)}}, \tag{4.4}$$

which have now to be used in inequalities (3.43) and (3.50). Here $K_2 > K_1$; $G_2 > G_1$.

It has been shown (HASHIN 1962) that (4.1) [equivalent expression, (38)] is an exact result for the bulk modulus of certain composite materials which may be described by a matrix of phase 'one' material in which spherical inclusions of phase 'two' material are distributed in a particular way. Analogously (4.2) is an exact result for the same case when the matrix is of phase 'two' material and the spherical inclusions of phase 'one' material. Since it has been shown here that (4.1) and (4.2) are always bounds on the effective bulk modulus they must be bounds also for this particular two phase material. It follows that these bounds are the most restrictive ones that can be given in terms of fractional volumes and phase moduli.

When the shear moduli of the two phases are equal the bounds (4.1) and (4.2) coincide, thus providing an exact result for this special case*. Equations (4.1) and (4.2) can be interpreted as exact results for two-phase materials with different phase bulk moduli but the same shear modulus (in one case G_1 and in the other G_2). This provides another proof that (4.1) and (4.2) are the most restrictive bounds that can be given in terms of phase moduli and volume fractions.

Expression (4.3) has also been previously obtained by HASHIN (1962) [equivalent expression (54)] by a heuristic argument. Since it has not proved possible to identify (4.3) or (4.4) with exact solutions the question whether or not (4.3) and (4.4) are the most restrictive bounds for G^* remains at present unanswered. However, previous results suggest that there are definite limits for improvement of the present bounds on the effective shear modulus.

For very small v_2, the lower bounds reduce to known results for effective elastic moduli of materials consisting of a matrix 1 in which a small amount of *spherical* non-homogeneities of material 2 is embedded. (ESHELBY 1957; HASHIN 1958, 1962). Analogous results are obtained from the upper bounds for small v_1.

The bounds (4.1)–(4.4) are generally farther apart than the bounds derived by HASHIN (1962). This is not surprising since the present two-phase material is more general and none of the phases is classified as either matrix or inclusions.

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*The authors are indebted to R. HILL for this observation. HILL has also derived the results for equal shear moduli and consequently, the bounds (4.1) and (4.2) by potential theory (private communication). Consequently the bounds (3.37), (3.38) have been examined and it has been shown that even these coincide when the shear moduli of all n phases are equal.

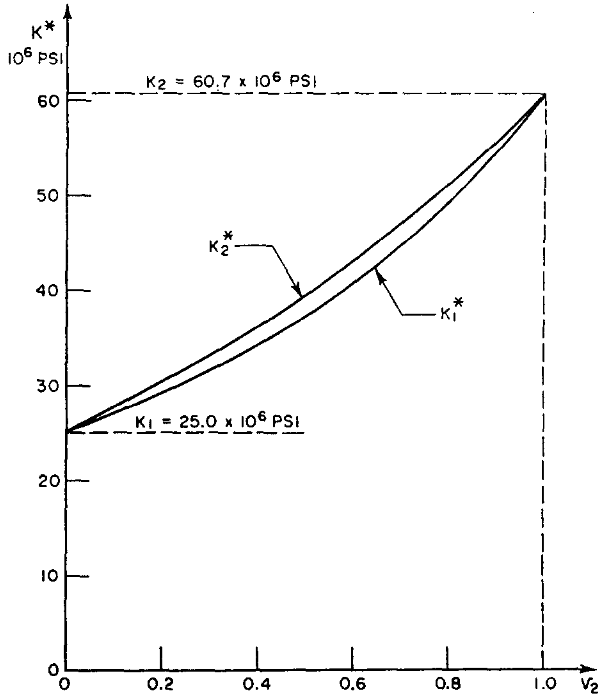

FIG. 1. Bounds for bulk modulus WC–Co alloy.

The bounds (4.1), (4.2), (4.3) and (4.4) have been numerically evaluated for a tungsten carbide–cobalt (WC-C_0) alloy for which experimental data are available (NISHIMATSU and GURLAND 1960). Here the moduli of the cobalt have been given the subscript 1 and those of the tungsten carbide the subscript 2. The bounds

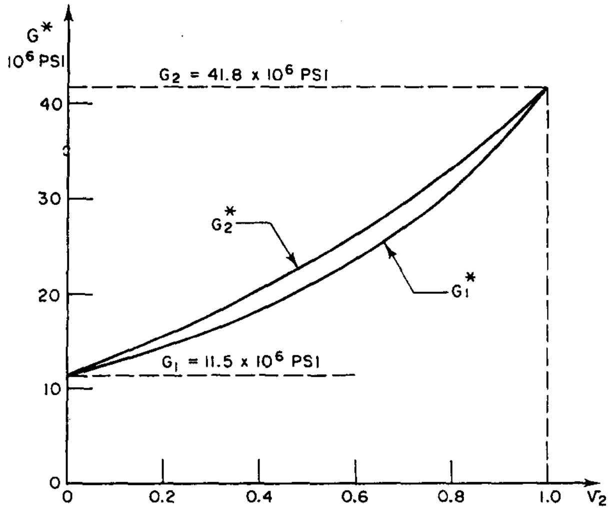

FIG. 2. Bounds for shear modulus Wc—Co alloy.

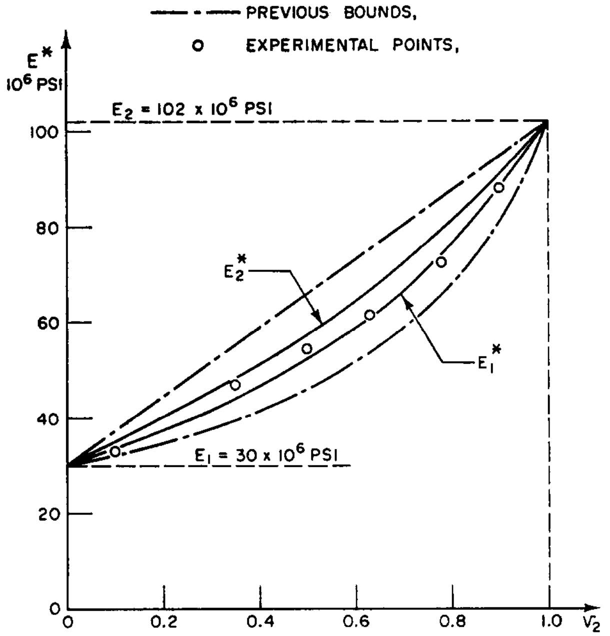

Fig. 3. Bounds for Young's modulus Wc–Co alloy.

for the bulk and shear moduli are shown in Figs. 1, 2. The bounds for the Young's modulus have been obtained by using the usual relation $E = 9KG/(3K + G)$. These are shown in Fig. 3 together with the experimental results and the bounds derived by Paul (1960). The comparison shows that the theory is in very good agreement with the experimental values of the effective Young's modulus and that the present bounds are a marked improvement.

It should be noted that the moduli of the tungsten carbide phase were quite large compared to those of the cobalt phase, the ratios being

$$\frac{K_2}{K_1} = 2{\cdot}4, \qquad \frac{G_2}{G_1} = 3{\cdot}6, \qquad \frac{E_2}{E_1} = 3{\cdot}4.$$

The bounds are nevertheless close together. This indicates that the bounds here obtained should give a good estimate for the effective moduli of two-phase materials for a variety of practical cases.

5. Discussion and Conclusions

The new variational principles, involving the polarization tensor, have proved to be a powerful tool for the analysis of multiphase materials. The present treatment employs isotropic fluctuating space functions, whose mean values only are known.

With such an approach one could, of course, try to use the differential equations of the theory of elasticity with space-dependent coefficients. This, however, seems to be exceedingly difficult and it is one of the advantages of the variational formulation that an extremum problem together with a boundary value problem for *homogeneous* elasticity can be substituted instead. Fortunately the full solution of the boundary value problem is not required and the interaction term U' can be evaluated on the basis of isotropic polarization distribution. Mathematical rigour of the method of evaluation of this term has yet to be established.

The variational principles for prescribed surface tractions, given by HASHIN and SHTRIKMAN (1962), can also be used in an analogous way. The analysis, however, would be more complicated because of the complexity of the subsidiary condition. It is believed that such an analysis would result in the same bounds, although this has not yet been checked.

The nature of the bounds obtained is such that the distance between them increases with increasing relative stiffness of one phase to others. In the extreme case of a rigid phase the upper bound will increase to infinity, whereas in the other extreme of an empty phase (cavities) the lower bound will decrease to zero. The question which naturally arises is : can the present bounds be improved ? So far the answer is known only for the effective bulk modulus of a two-phase material, where it has been shown that, given the present information (i.e. phase moduli and volume fractions), the bounds cannot be improved. It thus seems that improvement could be achieved only on the basis of further information, such as statistical details of phase distribution. This kind of information, however, is difficult to obtain in practice and it is at present not known how it could be used.

Although the foregoing conclusions, reached for the bulk modulus, cannot at present to be extended to the shear modulus, there seems to be little doubt that the shear modulus of multiphase materials is also indeterminate, and in fact more so than the bulk modulus, when only phase moduli and volume fractions are known.

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APPENDIX. EVALUATION OF U' BY FOURIER METHODS*

Consider a multiphase body of infinite extent. The problem can then be formulated as follows. Given

$$[L_0(\epsilon'_{ij}) + p'_{ij,j}] = t_{ij,j} = 0 \quad \text{in } V \tag{1}$$

where

$$\epsilon'_{ij} = \tfrac{1}{2}(u'_{i,j} + u'_{j,i}) \tag{2}$$

and

$$p'_{ij} = p_{ij} - \bar{p}_{ij}, \tag{3}$$

subject to the boundary condition

$$u'_i(S) = 0. \tag{4}$$

Evaluate

$$W = \tfrac{1}{2}\int p'_{ij}\,\epsilon'_{ij}\,dV. \tag{5}$$

and define U' by

$$U' = \lim \frac{W}{V}. \tag{6}$$

Define the three-dimensional complex Fourier transform Ψ of a function ψ by

$$\Psi(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{3/2}\int_{-\infty}^{\infty}\psi(\mathbf{x})\,e^{\iota\,\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x} \tag{7}$$

where **k** and **x** denote k_1, k_2, k_3 and x_1, x_2, x_3 respectively and $\iota = \sqrt{-1}$†.

Taking the transform of $t_{ij,j}$, (1) transforms to

$$k_j T_{ij} = 0 \tag{8}$$

where T_{ij} is the transform of t_{ij} and use has been made of the relation

$$\int_{(S)} t_{ij}\,n_j\,dS = 0. \tag{9}$$

Since t_{ij} is given, from (1) and (2) in terms of u'_i and p'_{ij}, the transform of (1), using (2) and (8), may be expressed in the form

$$(\lambda_0 + G_0)\,k_i k_m U_m + G_0\,\rho^2 U_i = -k_m P_{im}. \tag{10}$$

Here U_i and P_{ij} are the transforms of u'_i and p'_{ij} respectively, ρ is given by

$$\rho^2 = k_m k_m, \tag{11}$$

and (4) has been used.

Solving (10) for the U_i, the transform of $u'_{i,j}$, denoted by $U_{i,j}$, can be expressed in terms of U_i and accordingly in terms of P_{ij}. The result is

$$U_{i,j} = -\frac{P_{mr}\,k_m k_j\,[(\lambda_0 + 2G_0)\,\rho^2\delta_{ri} - (\lambda_0 + G_0)\,k_r k_i]}{G_0(\lambda_0 + 2G_0)\,\rho^4}. \tag{12}$$

From symmetry (5) remains unchanged when ϵ'_{ij} is exchanged by $u'_{i,j}$. By Parseval's theorem

$$2W = \int_{-\infty}^{\infty} P_{ij}(\mathbf{k})\,U^*_{i,j}(\mathbf{k})\,d\mathbf{k} \tag{13}$$

where the integration is over wave number space and * denotes a complex conjugate.

It is now assumed that each p_{ij} is an isotropic space function which, because of (3) and (3.6),

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*Compare treatment of a related magnetic problem by NEEL (1947), by means of Fourier series.

†For reference to Fourier transform rules, here and in the following, compare for example SNEDDON (1951).

is equivalent to the assumption of isotropy in the large of the multiphase material. As a consequence of this it may be shown that

$$P_{ij}(\mathbf{k}) = P_{ij}(\rho). \tag{14}$$

Introducing (12) into (13), and using (14), the integration in (13) is carried out using spherical co-ordinates in wave number space, given by

$$k_1 = \rho \cos \phi \sin \theta, \qquad k_2 = \rho \sin \phi \sin \theta, \qquad k_3 = \rho \cos \theta, \tag{15}$$

where

$$0 \leqslant \rho \leqslant \infty, \qquad 0 \leqslant \phi \leqslant 2\pi, \qquad 0 \leqslant \theta \leqslant \pi. \tag{16}$$

The result is

$$2W = -\frac{1}{G_0} \cdot \frac{4\pi}{3} \int_0^\infty P_{ij} P^*_{ij} \rho^2 \, d\rho + \frac{\lambda_0 + G_0}{G_0(\lambda_0 + 2G_0)} \cdot \frac{4\pi}{15} \int_0^\infty (P_{mm} P^*_{mm} + 2P_{ij} P^*_{ij}) \rho^2 \, d\rho. \tag{17}$$

Squaring both sides of (8) and integrating over space, using (3.6), one finds

$$\int p'_{ij} p'_{ij} \, dV = \sum_{r=1}^{r=n} p^r_{ij} p^r_{ij} V_r - \bar{p}_{ij} \bar{p}_{ij} V. \tag{18}$$

Using Parseval's theorem and (18) the right side of (17) can be completely expressed in terms of the p^r_{ij}. The result (3.15) is then obtained from (6) and (3.7).